Forest-fire models as a bridge between different paradigms in self-organized criticality

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We turn the stochastic critical forest-fire model introduced by Drössel and Schwabl [Phys. Rev. Lett. **69**, 1629 (1992)] into a completely deterministic threshold model. This model has many features in common with sandpile and earthquake models of self-organized criticality. Our deterministic forest-fire model exhibits in detail the same macroscopic statistical properties as the original Drössel-Schwabl model. We use the deterministic model to elaborate on the relation between forest-fire, sandpile, and earthquake models.

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I. INTRODUCTION

In this paper, we present the forest-fire (FF) model recast as a new deterministic, threshold model, which allows us to establish a bridge between the two main groups of *self-organized criticality* (SOC) systems. In particular, our FF model demonstrates that a nonconservative model can exhibit SOC even if no ''marginal synchronization'' is possible [1].

Several types of models of self-organized criticality exist [2,3]. The original cellular automaton models were defined by a deterministic and conservative updating algorithm with thresholds (barriers to activity) and stochastic driving [4,5]. A new variation of models was developed by Olami, Feder, and Christensen (OFC) [6] who realized that a nonconservative threshold model might remain critical if driven uniformly. The OFC model is completely deterministic except for a random initial configuration. In both types of model, the threshold is assumed to play a crucial role as a local rigidity that allows for a separation of time scales and, equally important, produces a large number of metastable states. The dynamics take the system from one of these metastable states to another. It is believed that separation of time scales and metastability are essential for the existence of scale invariance in these models.

A seemingly very different type of model was developed by Drössel and Schwabl (DS) [7]. No threshold appears explicitly in this model and the separation of time scales is put in by hand by tuning the rates of two stochastic processes that act as driving forces for the model. The DS FF is defined on a d-dimensional periodic square lattice. Empty sites are turned into ''trees'' with a probability p per site in every time step. A tree can catch fire stochastically when hit by ''lightning,'' with probability f each time step, or deterministically when a neighboring site is on fire. The model is found to be critical in the limit $p\rightarrow 0$ together with $f/p\rightarrow 0$. This model is a generalization of a model first suggested by Bak, Chen, and Tang (BCT) [8], which is identical to the DS model except that it does not contain the stochastic ignition by lightning. The BCT system is not critical in less than three dimensions, see [9–11]. A continuous variable, uniformly driven deterministic version [12] also shows regular behavior for low values of p [13]. Thus the introduction of the stochastic-lightning mechanism appeared to be necessary, at least in two dimensions, for the model to behave critically. A useful review can be found in [14].

In the present paper, we describe a transformation of the stochastic DS forest-fire model into a fully deterministic threshold system. This model is an extension of the recently introduced autoignition forest fire, a simple variation on the DS model [15]. As in that model, we find that all macroscopic statistical measures of the system are preserved. Specifically, we show that the three models have the same exponent for the probability density describing clusters of trees, similar probability densities of tree ages and, probably most unexpected, almost the same power spectrum for the number of trees on the lattice as a function of time. This latter is surprising, since even a small stochastic element in an updating algorithm is known to be capable of altering the power spectrum in a significant way [17]. We use characteristics of the new model to reveal links between the Bak, Tang, and Weisenfeld (BTW), OFC, and FF models of SOC hitherto unseen.

II. DEFINITION OF MODEL

The SOC FF can be recast into an autoignition model. This model is identical to the DS model, except that the spontaneous ignition probability f is replaced by an autoignition mechanism by which trees ignite automatically when their age T after inception reaches a value T_{max}. The increase in T is a *uniform* drive in the system. Choosing T_{max} suitably with respect to p gives a system with exactly the same behavior and statistical properties as the DS model [15]. Thus one stochastic driving process has been removed and a threshold introduced, while maintaining the SOC state; this model also displays explicitly the relationship between threshold dynamics and the separation of time scales so necessary for the SOC state.

We have found that the autoignition model can be turned into a completely deterministic critical model by eliminating the stochastic growth mechanism. The deterministic model (which we shall call the regen FF) is defined again on a periodic $2d$ lattice, of linear size L. Each cell is given an integer parameter T that increases by one each time step. If $T>0$, the cell is said to be occupied, otherwise it is empty (or regenerating). The initial configuration is a random distribution of T values and fires. Fires spread through nearest neighbors and the autoignition mechanism is again operative so that a tree catches fire when its $T=T_{max}$. However, in this

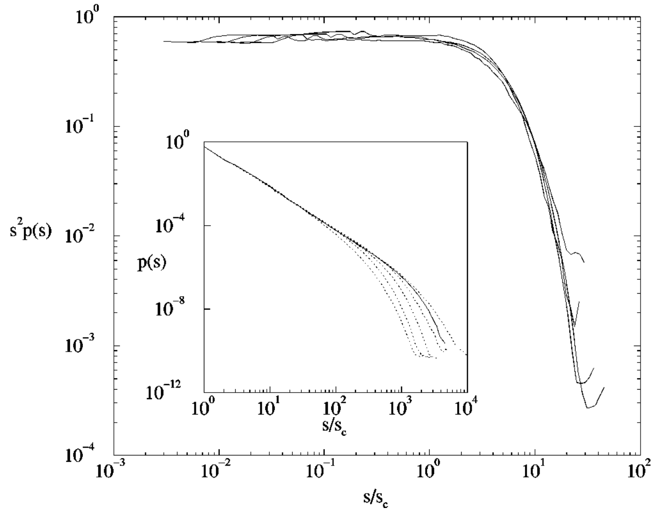

FIG. 1. Scaling plots of tree-cluster distributions for regen model ($L=1000$): $T_{max}=200$ and $t=T_{max}/T_{regen}=0.875,1.0$; $T_{max}=1000$ and $t=1.1$; $T_{max}=20\,000$ and $t=1.2$. Inset: Corresponding scaling of cutoff with t increasing left to right (dotted) and sample distribution for DS model with $p=0.001$, $f/p=0.01$ for comparison (solid).

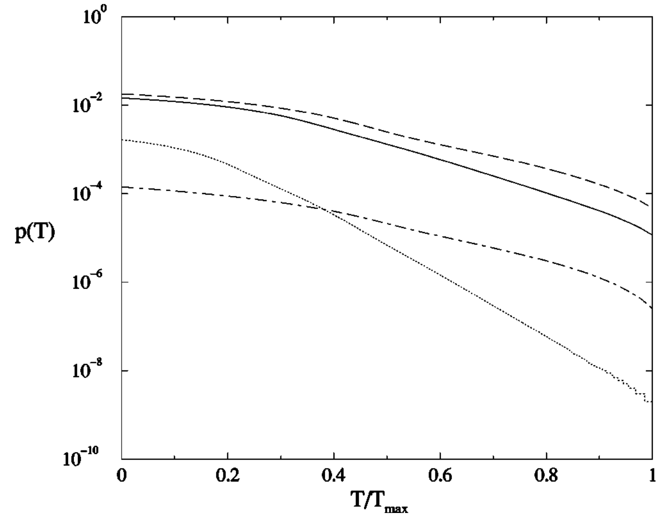

FIG. 2. Age profiles for DS ($p=0.001$, $f/p=0.01$, plotted with a T_{max} of 4000 for clarity, dotted), autoignition ($p=0.0001$, $T_{max}=20\,000$, and $T_{max}=24\,000$, dashed and solid) and regen ($T_{max}=24\,000$, $t=1.2$, dotted-dashed) models. All $L=1000$.

model when a tree catches fire the result is a decrement of T_{regen} from its T value. Note that when $T_{regen}<T_{max}$, a cell may still be occupied after it has been ignited. The parameters T_{max} and T_{regen} can be thought of as having a qualitatively reciprocal relationship with f and p, respectively, (in terms of the average ''waiting time'' for spontaneous ignition and tree regrowth), though this is less straightforward in the latter case because trees are not always burned down by fire. It is evident that T_{regen} also sets, and allows direct control of, the degree of dissipation of the T parameter in the system.

III. SIMULATION RESULTS

We now turn to a comparison between the statistical properties of the stochastic DS FF and the entirely deterministic regen model, with reference to the partly deterministic autoignition model.

First we consider the probability density $p(s)$ of the tree clusters sizes [18] simulated for different parameters for the different models. It is well known that the correlation length in the DS model [as measured by the cutoff s_c in $p(s)$] increases as the critical point is approached by decreasing p, f, and f/p [7]. There is a corresponding increase in the power-law regime for the cluster distribution in the autoignition model as p is decreased and T_{max} is increased [15]. The scaling behavior of the cutoff s_c is difficult to ascertain due to the limited range of data available, but seems to be of the form $\ln(S_c)\sim pT_{max}$, although we cannot exclude an algebraic dependence of the form $s_c\sim(pT_{max})^a$, with $a\simeq 6$. Figure 1 shows scaling plots for the regen model, and we see that here too the cutoff s_c scales with increasing ratio, $t=T_{max}/T_{regen}$. Note that the correlation length is very sensitive to the numerical value of t as compared to its dependence on f/p in the DS FF, and thus only a limited range of t is covered. For larger t (as with small f/p), the larger correlation length means that the lattice starts to become saturated with trees, reflected by distortion of the cluster size distribution, and the system size required to accomodate this becomes prohibitive. Nevertheless the covered range of t values corresponds to a wide range for the cutoff s_c as seen in Fig. 1.

We have approximately $\ln(s_c)\sim T_{max}$ though again the relation may be algebraic. The conclusion is that all three models approach a critical state described by the *same* power law $p(s)\sim s^{-\tau}$ with $\tau\simeq 2.04$ that within numerical accuracy agrees with previous simulation results [16,7].

Note that the regen model, being fully deterministic, is of course strictly periodic; but this ergodicity period (proportional to the total volume of phase space, $L^{d(T_{max}+T_{regen})}$) is extremely long and furthermore diverges with increasing T_{max}.

Let us now turn to the temporal characteristics of the models. In Fig. 2, we show that the probability distribution of the ages of the trees has a very similar form for all three models.

All are broad and exponential in character. Since it is a microscopic single-site property, it is not surprising that there is some variation between the models.

The DS FF exhibits a cutoff in the age distribution that is nearly as sharp as the cutoff in the two threshold models. This shows that the stochastic ignition process in the DS model, characterized by the lightning probability f, can be effectively replaced by the deterministic age threshold.

The collective temporal behavior is represented by the power spectrum of the time variation of the total number of trees on the lattice. In Fig. 3, these power spectra are shown for the DS and regen models (again, the power spectrum for the autoignition model is nearly identical).

Our most surprising result is that the deterministic regeneration model has nearly the same power spectrum as the two other models, particularly in the light of the differences in the age profiles above.

The equivalence between the three models allows us to think of the probabilistic growth and lightning in the DS FF model as effectively acting as thresholds. Qualitatively one

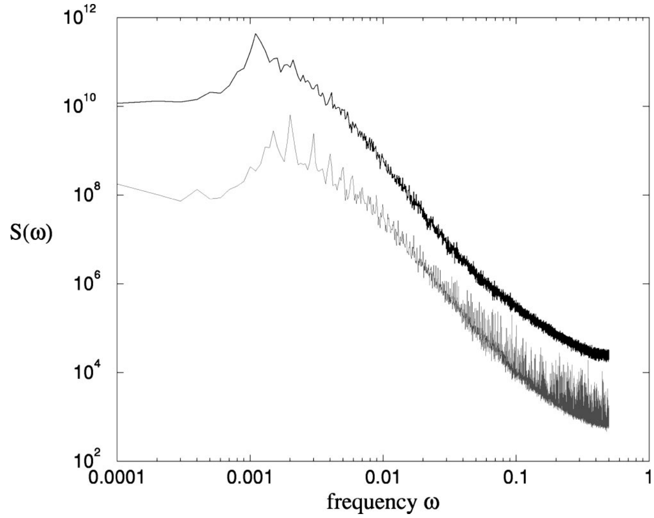

FIG. 3. Tree-density power spectra ($L=1000$) for DS ($f/p=0.01$, black) and regen ($T_{regen}/T_{max}=1.0$, gray) models.

can readily see that the probabilistic nature of the growth and the lightning can be interpreted as a kind of rigidity. Namely, an empty site has a rigidity against being turned into a tree described by $1/p$. A tree has rigidity against fire described by the fact that a tree only catches fire if nearest neighbor to a fire or when hit by lightning.

IV. DRIVING AND DISSIPATION: AN INTERPRETATION

The change in the mechanism for the renewal of the forest (from a probability for growth to a time for regeneration) and the resultant sandpilelike picture allows the identification of p with a dissipation parameter (in terms of the subtraction of T_{regen} on ignition) rather than as a driving parameter (see also [19]). This is quite contrary to the normally held and most obvious view—for the DS FF—that p is a driving parameter (creating trees in the system), and that if anything, f controls the dissipation (the complete combustion of trees into empty sites). Given this, we can speculate that it may be possible to equate the physical limits for critical behavior in the BTW sandpile:

$$h, h/\epsilon \to 0$$

(where h is the driving rate and ϵ the dissipation) and, recalling the qualitatively reciprocal relationships between f, p, T_{max}, and T_{regen} noted earlier, the DS and regen forest-fire models:

$$f, f/p \to 0, \quad \text{and} \quad 1/T_{max}, T_{regen}/T_{max} \to 0.$$

V. DISCUSSION AND CONCLUSION

We now discuss the relationship between the regen model presented above and other SOC models.

Our regen model is similar to the deterministic model introduced by Chen, Bak, and Jensen [12]. The crucial difference however, is that in the previous model the ratio T_{regen}/T_{max} — which must be decreased to move closer to the critical point and obtain scale free behavior — is effectively held fixed at a finite value, and hence the model does not allow one to truly approach the critical state.

The regen model has several features in common with the sandpile and earthquake models. It is similar to both sets of models in that the intrinsic dynamics is entirely deterministic and controlled by thresholds. The model is uniformly driven (in the T parameter) like the OFC earthquake model [6], and moreover, our deterministic FF model is genuinely nonconservative. It is worth noting that distributing the increase in T randomly in a limited number of portions (rather than equally across all trees), each time step was found to destroy the criticality as the size of the portions increased. In one important respect, our model is more similar to the BTW sandpile model than to the OFC model. Namely, when a site suffers relaxation (a tree catches fire), a fixed amount T_{regen} is subtracted from the dynamical variable of that site. The same happens in the BTW model. In the OFC model, on the other hand, the dynamical variable of a relaxing site is reset to zero. This property has been argued to allow for a marginal synchronization in the model and hence to be responsible for the OFC model's ability, in contrast to the BTW model, to remain critical even in the nonconservative regime [1]. Seen in this context, the deterministic FF model presented here constitutes a very interesting mix of features from the BTW and OFC models. Our regen FF model is nonconservative, uniformly driven, and, though the microscopic update does not support a marginal synchronization, nevertheless the model does exhibit the same scale free behavior as the DS FF.

The direct link established above between the SOC behavior of the BTW, OFC, and DS FF models, each of which are commonly assumed to be representative of different and distinct types of SOC models, is a step toward a unification of the physical origins of criticality in these systems (for related mean-field discussion, see [19]).

The main difference between the deterministic FF model and the sandpile and earthquake models is that the dynamical variable T is *not* transported to neighboring sites when a site relaxes and that the threshold exists only for the initiation and not the propagation of avalanches. This difference can be summarized as the FF model being a model of two coupled fields, fires and trees, whereas the sandpile and earthquake models contain one self-coupled field, the energy of a site.

Another difference is that the thresholds of the deterministic FF model must be tuned (to infinity) for the model to approach the critical regime. The reason for this is that the thresholds relate directly to the rate of driving in the model. The sandpile and earthquake models are different in that the SOC limit of slow driving can be reached without a tuning of the thresholds.

The relationship between deterministic and stochastic SOC models has been studied in the context of the BTW sandpile model in a paper by Wiesenfeld, Theiler, and McNamara [20]. They replaced the spatial randomness in the dropping of sand grains by a seeding of sand grains at the central site of the lattice only. Without a stochastic element in the driving, the model becomes entirely deterministic but *remains* critical. Wiesenfeld, Theiler, and McNamara then applied a dynamical systems point of view and were able to conclude that the criticality survives due to the coexistence of many periodic attractors that lead them to picture the critical state as the union of many coexisting orbits. This picture does not directly apply to our deterministic FF model. Incrementing the T value of a single tree only can only lead to a trivial dynamics. The driven site will ignite when its T value reaches T_{max} and a fire will spread through the trees spatially connected to the driven site. When this has happened, the dynamics of the lattice reduces to the trivial oscillation of the T value of the driven site between T_{max}-T_{regen} and T_{max}. As mentioned before, the regen model must be driven globally and the global drive has to be uniform.

Finally, we note that the regen model is deterministic and critical with periodic boundary conditions. This is in contrast to deterministic versions of the BTW and OFC that become noncritical when periodic boundary conditions are applied. The regen model is without external stochastic driving (unlike the DS model) and in this sense the regen model can be said to be completely self-contained in its dynamics. As far as we know, the regen model is the only deterministic uniformly driven model that remains critical when periodic boundary conditions are applied.

In summary, we have demonstrated that the stochastic Drössel-Schwabl forest-fire model can be turned into a deterministic threshold model without changing any of the collective statistical measures of the system in a significant way. The model illuminates greatly the relationship between different types of SOC models.

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